

The renormalized ϕ_4^4 -trajectory by perturbation theory in the running coupling

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Abstract

We compute the renormalized trajectory of ϕ_4^4 -theory by perturbation theory in a running coupling. We introduce an iterative scheme without reference to a bare action. The expansion is proved to be finite to every order of perturbation theory.

1 Introduction

In Wilson's renormalization group [W71, WK74] ultraviolet and infrared limit stand for an infinite iteration of block spin transformations. Consider for instance the ultraviolet limit of an asymptotically free model at weak coupling. There the point is to keep couplings under control which grow under a block spin transformation. Such couplings are called relevant. In weakly coupled models they can be identified by power counting. Renormalization of a bare action amounts to sending it through an increasing number of block spin transformations. The image is the renormalized action. For this limit to exist the bare couplings need to be tuned as the number of block spin transformations is increased. This renormalization scheme has been beautifully implemented both within and beyond perturbation theory. Let us mention the work of Gallavotti [G85], Gawedzki and Kupiainen [GK77, GK84], and Polchinski [P84] as a guide to the extensive literature. The renormalized actions are located on a low dimensional curve, parametrized by renormalized couplings. In the case of ϕ^4 -theory we can consider a one-dimensional subspace and speak of a renormalized trajectory. The idea is to arrange the renormalization group flow generated from the bare action as to converge to this renormalized trajectory in the process of renormalization. The renormalized trajectory of an asymptotically free model is pictured as unstable manifold of a trivial fixed point.

Although this picture has been behind block spin renormalization since the very beginning it has not yet been formalized to an approach free of bare action. This paper is a contribution to fill this gap. It extends the analysis begun in [WX94, RW95] for the hierarchical approximation to the ϕ^4 -model with discrete momentum space renormalization group. The renormalized trajectory is here defined as a curve in the space of effective actions which passes through the trivial fixed point and whose tangent at the trivial fixed point is a ϕ^4 -vertex. The dynamical principle which proves to be sufficiently strong to determine this curve at least to all orders of perturbation theory is stability under the renormalization group. With stability we mean here that the curve is left invariant as a set in the space of effective actions under a block spin transformation. A renormalized action always comes together with a sequence of descendants generated by further block spin transformations. Even in the case of a discrete renormalization group this sequence proves to consist of points on a continuous curve in the space of effective actions which is stable under a block spin transformation. It is the computation of this curve in a vicinity of the trivial fixed point we address.

Given a block spin transformation, we may distinguish between the following different renormalization problems. The first problem is an initial value problem, the analysis of the renormalization group flow started from a particular bare action. The second problem is a mixed boundary value problem, where the relevant parameters are prescribed on a lower scale, the irrelevant parameters on a higher one. The first problem is appropriate for the infrared limit of a Euclidean field. The second is appropriate

for its ultraviolet limit. The third problem is the question of fixed points. In this paper we consider a generalization of the fixed point problem. We will look for interactions which remain invariant up to a (one dimensional) flow of a coupling parameter. A requirement of finiteness will substitute for boundary data. Of course the problems are interrelated. In particular the mixed boundary value problem is a method to obtain a solution of our generalized fixed point problem in a scaling limit.

The result is an iterative form of perturbation theory in a running coupling. Its closest relative in the literature is the renormalized tree expansion of Gallavotti and collaborators [G85, GN85]. Our expansion will however not be organized in trees. Furthermore, it will from the very beginning be free of divergencies piled up in standard perturbation theory. Surprisingly it will allow to treat relevant and irrelevant couplings on the same footing. It will involve neither bare couplings nor renormalization conditions in the original sense. The expansion will be presented for the ϕ^4 -trajectory in four dimensions. Most of the analysis, in fact everything except for the treatment of the wave function term, works in arbitrary dimensions. We therefore leave D as a parameter in the equations. Thereby dimension dependence of scale factors is exhibited. The three dimensional case requires a modification which will be explained elsewhere. See [RW95] for a treatment of its hierarchical approximation.¹

To make contact with the physical world one has to supply one more piece of information. One has to assign a scale to a point on the renormalized trajectory. In the presentation of this paper we will maintain a unit scale throughout the computation. In view of asymptotic freedom the four dimensional trajectory is suited for the infrared limit of massless ϕ^4 -theory at positive coupling. It is also suited for the ultraviolet limit of massless ϕ^4 -theory at negative coupling, as been promoted by Gawedzki and Kupiainen [GK85]. In this paper we will restrict our attention to the effective action. Its relation with Schwinger functions is discussed for instance in [BG95]. The iterative solution of our equations can also be viewed as an improvement program organized in powers of a running coupling.

2 Renormalization group

The below analysis will be done in terms of a discrete momentum space renormalization group transformation. A number of applications of which is discussed for instance in recent lectures [BG95] by Benfatto and Gallavotti. Let us consider the following discrete block spin transformation \mathcal{R} on some space of interactions $V(\phi)$ of a real scalar field ϕ on Euclidean space \mathbb{R}^D . Let \mathcal{R} be composed of a Gaussian fluctuation integral with covariance Γ and mean ψ with a dilatation \mathcal{S} of the background field ψ .

¹In three dimensions the expansion parameter has a non zero scaling dimension. As a consequence the flow of the mass term shows up a second order correction proportional to the logarithm of the running coupling. This type of corrections requires of a double expansion in the running coupling and its logarithm.

Let the fluctuation covariance be defined by

$$\hat{\Gamma}(p) = \frac{1}{p^2} (\hat{\chi}(p) - \hat{\chi}(Lp)), \quad (1)$$

where $\hat{\chi}(p)$ is a momentum space cutoff function. Its purpose is to make $\hat{\Gamma}(p)$ decrease fast outside a momentum slice $L^{-1} < |p| < 1$ set by a scale parameter $L > 1$. A convenient choice is the exponential cutoff

$$\hat{\chi}(p) = e^{-p^2}. \quad (2)$$

It will be used in the following. Other choices however work as well, for instance Pauli-Villars regularization. Then (1) defines a positive operator on the subspace of $L_2(\mathbb{R}^D)$ consisting of functions $f(x)$ with zero mode $\hat{f}(0) = 0$. Let $d\mu_\Gamma(\zeta)$ be the associated Gaussian measure on field space. Recall its basic property

$$\int d\mu_\Gamma(\zeta) e^{(\zeta, f)} = e^{\frac{1}{2}(f, \Gamma f)} \quad (3)$$

and consult Glimm and Jaffe [GJ87] for further information. Let the fluctuation integral then be given by the average of the Boltzmann factor $Z(\phi) = \exp(-V(\phi))$ with respect to $d\mu_\Gamma(\zeta)$, shifted by an external background field ψ . Let us introduce the notation

$$\langle Z \rangle_{\Gamma, \psi} = \int d\mu_\Gamma(\zeta) Z(\psi + \zeta) \quad (4)$$

for this average. We can think of the momentum slice $L^{-1} < |p| < 1$ as a portion of momentum space degrees of freedom which is integrated out. The integration of another portion is prepared for by a dilatation of the background field. Let this dilatation be given by

$$\mathcal{S}\psi(x) = L^{1-\frac{D}{2}} \psi\left(\frac{x}{L}\right). \quad (5)$$

The exponent $\sigma = 1 - D/2$ is the scaling dimension of a free massless scalar field. Anomalous rescaling will not be considered here. Non-anomalous rescaling applies (at least) to small perturbations of a free massless field. The renormalization group transformation is then defined by (4) composed with (5). The following analysis will be done in terms of the potential $V(\phi)$. The method will be perturbation theory. The matter of stability bounds on $Z(\phi)$ will not be addressed here. The renormalization group transformation for the potential then reads

$$\mathcal{R}V(\psi) = -\log \left(\langle \exp(-V) \rangle_{\Gamma, \mathcal{S}\psi} \right). \quad (6)$$

We will restrict our attention to even potentials $V(-\phi) = V(\phi)$. The transformation (6) preserves this property. Potentials differing by a field independent constant will be identified. $V(\phi)$ can for instance be normalized such that $V(0) = 0$. To maintain

normalization, (6) then should be supplemented by a subtraction of $\mathcal{R}V(0)$. Technically this constant is proportional to the volume, infinite in infinite volume. Eq. (6) therefore requires an intermediate volume cutoff to make sense. We will wipe this technicality under the carpet, keep (6) as it stands, and ignore field independent constants. When we perform a renormalization group transformation we will speak of $V(\phi)$ as bare and of $\mathcal{R}V(\phi)$ as effective or renormalized potential. It should however be kept in mind that only degrees of freedom in one momentum slice are integrated out in a single renormalization group step. ζ will be called fluctuation field and ψ background or block spin field. The term potential is sometimes reserved for local interactions. Here potential will be used synonymous with full interaction including nonlocal interactions generated by the renormalization group. The block scale L will be kept fixed in the following. It should not be confused with a full momentum space cutoff of a Euclidean field. A typical value of L is two. A full cutoff could be an N 'th power of L .

3 Trivial fixed point

The renormalization group transformation (6) has a trivial fixed point $V_*(\phi) = 0$. This fixed point is the free massless scalar field. Eq. (6) has in fact been designed upon a momentum space decomposition of a free massless field. The linearized renormalization group transformation at this fixed point is given by a Gaussian expectation value

$$D_{V_*} \mathcal{R}\mathcal{O}(\psi) = \langle \mathcal{O} \rangle_{\Gamma, \mathcal{S}\psi}, \quad (7)$$

shifted by a rescaled external field. It is diagonalizable. The eigenvectors are normal ordered products. We will represent the potential in terms of normal ordered products. Let us therefore briefly recall some basic facts about normal ordering. Normal ordered products with normal ordering covariance v are generated by

$$: e^{(\phi, f)} :_v = e^{(\phi, f) - \frac{1}{2}(f, v f)}. \quad (8)$$

The linearized renormalization group for this generating function is an exercise in Gaussian integration (3). The result is

$$\int d\mu_{\Gamma}(\zeta) : e^{(\mathcal{S}\psi + \zeta, f)} :_v = : e^{(\psi, \mathcal{S}^T f)} :_{\mathcal{T}v}. \quad (9)$$

The generating function (8) is preserved up to a (transposed) dilatation

$$\mathcal{S}^T f(x) = L^{1+D/2} f(Lx) \quad (10)$$

of the source and a linear transformation

$$\mathcal{T}v = \mathcal{S}^{-1}(v - \Gamma)(\mathcal{S}^{-1})^T \quad (11)$$

of the normal ordering covariance. This linear transformation generates a flow of normal ordering. It can be thought of as a residual renormalization group flow taking place besides more interesting dynamical effects. It has a line of fixed points

$$\hat{v}(p) = \frac{1}{p^2} (\hat{\chi}(p) - C) \quad (12)$$

parametrized by C . Let us select the point $C = 0$ as normal ordering covariance in the following, a massless covariance with unit ultraviolet cutoff. Since it remains invariant under (11) it can be safely suppressed in the notation. It follows that

$$\mathcal{O}(\phi) = \int d^D x_1 \cdots d^D x_n : \phi(x_1)^{m_1} \cdots \phi(x_n)^{m_n} : \quad (13)$$

is an eigenvector of the linearized renormalization group (7) with eigenvalue L^σ . The exponent is $\sigma = nD + (m_1 + \cdots + m_n)(1 - D/2)$, the scaling dimension of (13). A prominent member of this family is the ϕ^4 -vertex

$$\mathcal{O}(\phi) = \int d^D x : \phi(x)^4 : \quad (14)$$

with scaling dimension $4 - D$. It is therefore called relevant in $D < 4$, marginal in $D = 4$, and irrelevant in $D > 4$ dimensions. General eigenvectors are given by homogeneous kernels in real or momentum space and also involve derivatives of fields. See the review [W76] by Wegener.

4 Perturbation theory

In a vicinity of the trivial fixed point the transformation (6) can be computed by means of perturbation theory. The perturbation expansion for the effective potential reads

$$\mathcal{R}V(\psi) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \langle [V;]^n \rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (15)$$

The superscript T indicates truncated expectation values. Truncated expectation values are defined by

$$\left\langle \prod_{i=1}^n [\mathcal{O}_i;] \right\rangle^T = \left[\left(\prod_{i=1}^n \frac{\partial}{\partial \lambda_i} \right) \log \left\langle \exp \left(\sum_{i=1}^n \lambda_i \mathcal{O}_i \right) \right\rangle \right]_{\lambda_1 = \cdots = \lambda_n = 0}. \quad (16)$$

Notice that the truncated expectation values $\langle [V;]^n \rangle_{\Gamma, \mathcal{S}\psi}^T$ are the cumulants associated with the moments $\langle V^n \rangle_{\Gamma, \mathcal{S}\psi}$. The perturbation expansion (15) is therefore also known as cumulant expansion. Let us consider the situation when the potential comes in form of a power series

$$V(\phi|g) = \sum_{n=1}^{\infty} \frac{g^n}{n!} V^{(n)}(\phi) \quad (17)$$

of a coupling parameter g . The zeroth order interaction will always be $V^{(0)}(\phi) = 0$ in the following. In the bare perturbation expansion for a single renormalization group transformation the effective interaction is expanded again in the bare coupling. Let us introduce the notation

$$\mathcal{R}V(\psi|g) = \sum_{n=1}^{\infty} \frac{g^n}{n!} (\mathcal{R}V)^{(n)}(\psi) \quad (18)$$

for this expansion. The individual orders of perturbation theory are given by sums of truncated expectation values

$$(\mathcal{R}V)^{(n)}(\psi) = \sum_{m=1}^n \frac{(-1)^{m+1}}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} \langle V^{(l_1)}; \dots; V^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (19)$$

The sum over (l_1, \dots, l_m) is restricted to m -tupels in $\{1, \dots, n\}^m$ such that $l_1 + \dots + l_m = n$ and is finite. The multinomial coefficient is given by

$$\binom{n}{l_1 \dots l_{m-1}} = \frac{n!}{\prod_{k=1}^m l_k!} \quad (20)$$

with $l_m = n - l_1 - \dots - l_{m-1}$. To lowest orders we find the following explicit expressions for the effective interactions

$$(\mathcal{R}V)^{(1)}(\phi) = \langle V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}, \quad (21)$$

$$(\mathcal{R}V)^{(2)}(\phi) = \langle V^{(2)} \rangle_{\Gamma, \mathcal{S}\psi} - \langle V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T, \quad (22)$$

$$(\mathcal{R}V)^{(3)}(\phi) = \langle V^{(3)} \rangle_{\Gamma, \mathcal{S}\psi} - 3 \langle V^{(2)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T + \langle V^{(1)}; V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (23)$$

Let us have a closer look at two contributions to (19). The highest order bare interaction appearing to order n is $V^{(n)}(\phi)$. It contributes a term $\langle V^{(n)} \rangle_{\Gamma, \mathcal{S}\psi}$. It is thus transformed according to the linearized renormalization group (7). The perturbative corrections to the linearized renormalization group depend on lower orders $V^{(m)}(\phi)$ with $1 \leq m \leq n-1$ only. The first order $V^{(1)}(\phi)$ contributes by itself a term $(-1)^{n+1} \langle [V^{(1)};]^n \rangle_{\Gamma, \mathcal{S}\psi}^T$ to $(\mathcal{R}V)^{(n)}(\psi)$. These interactions need to be carried along immediately to order n when some first order interaction enters the game. In a minimal scheme no further interactions would be introduced to this order.

(19) leaves us with an expansion for the effective potential in terms of the bare coupling. This expansion is not appropriate for an iteration of renormalization group transformations. The appropriate expansion is an expansion in powers of the effective coupling. This requires a reorganization of (19) in powers of the effective coupling. Let

this effective coupling come as a power series

$$\beta(g) = \sum_{n=1}^{\infty} \frac{g^n}{n!} b_n \quad (24)$$

in the bare coupling. The function $\beta(g)$ is not a Callan-Symanzik β -function in a literal sense but its block spin analogue. The inverse reorganization goes as follows. Let the effective potential be given as a power series (17) of the effective coupling. Then we can substitute (24) and expand the result in powers

$$V(\psi|\beta(g)) = \sum_{n=1}^{\infty} \frac{g^n}{n!} (V \circ \beta)^{(n)}(\psi) \quad (25)$$

of the bare coupling. The coefficients in this reorganized expansion (25) are given by

$$(V \circ \beta)^{(n)}(\psi) = \sum_{m=1}^n \frac{1}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} b_{l_1} \dots b_{l_m} V^{(m)}(\psi). \quad (26)$$

in terms of the coefficients of (17) and (24). The zeroth order of (24) is put to $b_0 = 0$. To lowest orders (26) is given by

$$(V \circ \beta)^{(1)}(\psi) = b_1 V^{(1)}(\psi), \quad (27)$$

$$(V \circ \beta)^{(2)}(\psi) = b_2 V^{(1)}(\psi) + (b_1)^2 V^{(2)}(\psi), \quad (28)$$

$$(V \circ \beta)^{(3)}(\psi) = b_3 V^{(1)}(\psi) + 3b_1 b_2 V^{(2)}(\psi) + (b_1)^3 V^{(3)}(\psi). \quad (29)$$

This combinatorial exercise completes the setup for perturbation theory.

If we would iterate (19) we would end up with the tree expansion of Gallavotti [G85]. See [FHRW88] for a pedagogical account of this well developed technology. We will not organize perturbation theory in terms of trees. Also we will not start with a bare perturbation expansion but directly attack the renormalized series. Let us conclude this section with the remark that the effective potential (15) is the generating function of free propagator amputated connected Green's functions with vertices (17) and propagator Γ . The terms in (19) can therefore be expressed in terms of connected Feynman diagrams.

5 ϕ^4 -trajectory

Let us pose the following renormalization problem. We seek a potential $V(\phi|g)$ depending on a running coupling g with the following properties. ("running" will be explained below.)

1) $V(\phi|g)$ is a power series

$$V(\phi|g) = \sum_{n=1}^{\infty} \frac{g^n}{n!} V^{(n)}(\phi) \quad (30)$$

in the coupling parameter g .

The zeroth order is here $V^{(0)}(\phi) = 0$. We will treat of (30) as a formal perturbation of the trivial fixed point. The important question of summability of (30) will not be addressed.

2) The first order in (30) is a ϕ^4 -vertex

$$V^{(1)}(\phi) = \frac{1}{4!} \int d^D x : \phi(x)^4 : . \quad (31)$$

The n 'th order in (30) is a polynomial

$$V^{(n)}(\phi) = \sum_{m=1}^{n+1} \frac{1}{2m!} \int dx_1 \dots dx_{2m} V_{2m}^{(n)}(x_1, \dots, x_{2m}) : \phi(x_1) \dots \phi(x_{2m}) : \quad (32)$$

in the field ϕ .

Let us imagine a bare theory consisting purely of a first order vertex (31), possibly shouldered by second order mass and wave function counterterms. The effective interactions generated in the course of its renormalization group flow will not remain of this simple form. The first renormalization group step already leaves us with an infinite set of higher order vertices, among which are for example second order mass and wave function terms. A main theme of this paper is to add all these higher order interactions to the bare theory from the beginning since they will anyway be generated once (31) enters the theory. In a minimal scheme no other vertices are introduced than those enforced by the presence of (31). We can then think of (31) as the germ of the theory. Other trajectories emerging from the trivial fixed point are of interest as well, for instance the ϕ^6 -trajectory. There the first order is a ϕ^6 -vertex. Our leitmotiv here is to keep the effective interaction as minimal as possible. The highest connected vertex generated from n first order ϕ^4 -vertices has $2n + 2$ legs. Higher vertices will appear at this order only if they are introduced by hand. (32) excludes this option. This form of potential iterates through the renormalization group. Field independent terms are discarded. We consider even powers of fields only.

4) The kernels in (32) are Euclidean invariant distributions. They are given by Fourier integrals

$$V_{2m}^{(n)}(x_1, \dots, x_{2m}) = \int \frac{d^D p_1}{(2\pi)^D} \dots \frac{d^D p_{2m}}{(2\pi)^D} e^{i \sum_{l=1}^{2m} p_l x_l} (2\pi)^D \delta \left(\sum_{l=1}^{2m} p_l \right) \hat{V}_{2m}^{(n)}(p_1, \dots, p_{2m}). \quad (33)$$

With the δ -function removed their Fourier transforms are symmetric \mathcal{C}^∞ -functions on momentum space $\mathbb{R}^D \times \dots \times \mathbb{R}^D$. They satisfy the bounds

$$\left\| p^\alpha \frac{\partial^{|\alpha|}}{\partial p^\alpha} \widehat{V}_{2m}^{(n)} \right\|_{\infty, \epsilon} = \sup_{(p_1, \dots, p_{2m}) \in \mathcal{M}_{2n}} \left\{ \left| p^\alpha \frac{\partial^{|\alpha|}}{\partial p^\alpha} \widehat{V}_{2m}^{(n)}(p_1, \dots, p_{2m}) \right| e^{-\epsilon(|p_1| + \dots + |p_{2m}|)} \right\} < \infty. \quad (34)$$

for any $\epsilon > 0$. Here \mathcal{M}_{2m} denotes the subset of momenta $(p_1, \dots, p_{2m}) \in \mathbb{R}^D \times \dots \times \mathbb{R}^D$ with $p_1 + \dots + p_{2m} = 0$. Furthermore, $\alpha = (\alpha_{i,\mu}) \in \mathbb{N}^{2mD}$ is a multi-index and $|\alpha| = \sum_{i,\mu} \alpha_{i,\mu}$.

The kernels in (33) are the unknowns in this approach. (34) is meant as requirement on what kind of kernels we will hold look out for. This condition of finiteness substitutes for boundary data which is necessary for instance in [G85] and [P84]. On the practical side we want to be certain that our perturbation theory is finite to every order. The $L_{\infty, \epsilon}$ -norm in momentum space is a convenient but not the only possible criterion. See [FHRW88] for further inspiration. The large momentum bound implied by (34) is rather wasteful. The accurate large momentum behavior is polynomial in powers and logarithms of momenta. Their origin is a Taylor expansion of the quadratic and quartic kernel in four dimensions. The Taylor expansion is compelled by power counting. The remainders are irrelevant. The norm estimates present bounds on these irrelevant remainder terms. The Taylor coefficients are of course also required to be finite. This is understood when we speak of smooth functions on momentum space. Smoothness is a little more than what is needed. Existence of required derivatives at zero momentum (or some general subtraction point) together with $L_{\infty, \epsilon}$ -bounds on the remainders is fully sufficient. In the ultraviolet problem with exponential cutoff we can afford the luxury of smoothness. Euclidean invariance reduces the Taylor coefficients in four dimensions to a mass term, a wave function term, and the ϕ^4 -vertex. Note that (6) respects Euclidean invariance. It is therefore broken only if we break it by hand. Note also that translation invariance by itself affects momentum space power counting.

5) The four point kernel (33) is

$$\widehat{V}_4^{(n)}(0, 0, 0, 0) = \delta_{n,1} \quad (35)$$

at zero momentum.

The zero momentum condition (35) is part of the definition of the expansion parameter g . If g is traded for another parameter $g'(g) = g + O(g^2)$ then (35) is not true anymore. There is nothing wrong with a redefinition of g , although it looks unnecessary to introduce a ϕ^4 -vertex at some higher order when it is already present to first order.

6) There exists a power series

$$\beta(g) = \sum_{n=1}^{\infty} \frac{g^n}{n!} b_n \quad (36)$$

such that

$$\mathcal{R}V(\psi|g) = V(\psi|\beta(g)) \quad (37)$$

to every order in g .

This condition is the core of our approach. It says that $V(\phi|g)$ remains of the same form under the renormalization group up to flow of g . It is therefore called running coupling. (37) will prove to be strong enough to determine both $V(\phi|g)$ and $\beta(g)$ to every order in g . In $D = 4$ dimensions this scheme will require a modification due to wave function renormalization. We will need to introduce a first order wave function term. But let us postpone this modification for the moment. A potential with the property (37) will be said to scale.

It is amusing to think of the solution $V(\phi|\cdot) : g \mapsto V(\phi|g)$ as a parametrized curve in interaction space. Anticipating a future renormalization group geometry we can then say that: 1) $V(\phi|\cdot)$ visits the trivial fixed point at $g = 0$. 2) The tangent to $V(\phi|\cdot)$ at $g = 0$ is the ϕ^4 -vertex. 3) $V(\phi|\cdot)$ is stable under the renormalization group as a set in interaction space. We call this curve ϕ^4 -trajectory.

6 Scaling equations

Expanding both sides of (37) in powers of g , we obtain a system of scaling equations for the perturbations series (30) and (36). This system is organized into a recursion relation. From (21), (22), (23) and (27), (28), (29) we find to lowest orders equations of the form

$$\langle V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi} - b_1 V^{(1)}(\psi) = 0, \quad (38)$$

$$\langle V^{(2)} \rangle_{\Gamma, \mathcal{S}\psi} - (b_1)^2 V^{(2)}(\psi) = b_2 V^{(1)}(\psi) + \langle V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T, \quad (39)$$

$$\begin{aligned} \langle V^{(3)} \rangle_{\Gamma, \mathcal{S}\psi} - (b_1)^3 V^{(3)}(\psi) &= b_3 V^{(1)}(\psi) + 3b_1 b_2 V^{(2)}(\psi) + 3 \langle V^{(1)}; V^{(2)} \rangle_{\Gamma, \mathcal{S}\psi}^T - \\ &\quad \langle V^{(1)}; V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T. \end{aligned} \quad (40)$$

To first order, scaling requires $V^{(1)}(\psi)$ to be an eigenvector of the linearized renormalization group, and b_1 to be the eigenvalue. This is the case for the ϕ^4 -vertex with the familiar eigenvalue $b_1 = L^{4-D}$. The first order equation is special in that it is homogeneous. To higher orders we meet inhomogeneous linear equations. Their general form for $n \geq 2$ is

$$\begin{aligned} \langle V^{(n)} \rangle_{\Gamma, \mathcal{S}\psi} - (b_1)^n V^{(n)}(\psi) &= \\ b_n V^{(1)}(\psi) + (-1)^n \langle V^{(1)}; \dots; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T &+ \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} \end{aligned}$$

$$\sum_{m=2}^{n-1} \frac{1}{m!} \left(b_{l_1} \cdots b_{l_m} V^{(m)}(\psi) + (-1)^m \left\langle V^{(l_1)}; \dots; V^{(l_m)} \right\rangle_{\Gamma, \mathcal{S}}^T \right). \quad (41)$$

We give the name $L^{n(4-D)} K^{(n)}(\psi)$ to the right hand side of (41). It is determined by the effective potential $V^{(m)}(\psi)$ to lower orders $1 \leq m \leq n-1$. Eq. (41) poses an inhomogeneous linear problem for $V^{(n)}(\psi)$. Its left hand side is diagonalized by normal ordering. Insert the expansion (32) in normal ordered powers of fields to obtain

$$\begin{aligned} \left\langle V^{(n)} \right\rangle_{\Gamma, \mathcal{S}\psi} - (b_1)^n V^{(2)}(\psi) = \\ \sum_{m=1}^{n+1} \frac{1}{(2m)!} \int d^D x_1 \cdots d^D x_{2m} : \psi(x_1) \cdots \psi(x_{2m}) : \\ \left(L^{m(D+2)} V_{2m}^{(n)}(Lx_1, \dots, Lx_{2m}) - L^{n(4-D)} V^{(n)}(x_1, \dots, x_{2m}) \right). \end{aligned} \quad (42)$$

The linearized renormalization group thus performs a scale transformation of a kernel.² The difference of exponents defines an order dependent real space power counting

$$\sigma(m, n) = m(D+2) - n(4-D). \quad (43)$$

The right hand side of (41) can now be expanded in the same manner. We introduce the notation

$$K^{(n)}(\psi) = \sum_{m=1}^{n+1} \frac{1}{(2m)!} \int d^D x_1 \cdots d^D x_{2m} : \psi(x_1) \cdots \psi(x_{2m}) : K_{2m}^{(n)}(x_1, \dots, x_{2m}) \quad (44)$$

for this expansion in normal ordered powers of fields. From (42) and (44) together we then deduce the equations

$$L^{\sigma(m,n)} V_{2m}^{(n)}(Lx_1, \dots, Lx_{2m}) - V_{2m}^{(n)}(x_1, \dots, x_{2m}) = K_{2m}^{(n)}(x_1, \dots, x_{2m}). \quad (45)$$

The problem of renormalized perturbation theory has thus been reduced to the solution of (45). Recall that the right hand side is determined by vertices of lower orders and cutoff propagators. The kernels are distributions on copies of real space. Fourier transformation turns (45) into

$$L^{\hat{\sigma}(m,n)} \hat{V}_{2m}^{(n)}\left(\frac{p_1}{L}, \dots, \frac{p_{2m}}{L}\right) - \hat{V}_{2m}^{(n)}(p_1, \dots, p_{2m}) = \hat{K}_{2m}^{(n)}(p_1, \dots, p_{2m}). \quad (46)$$

Here the δ -function due to translation invariance has been removed. Eq. (46) is the main dynamical equation in this approach. The task is to search for finite solutions in the sense of (34). In momentum space we find an order dependent power counting

$$\hat{\sigma}(m, n) = D - m(D-2) - n(4-D). \quad (47)$$

²Homogeneous kernels therefore give eigenvectors of the linearized transformation.

In the sequel the attributes relevant, marginal, and irrelevant will be used to distinguish whether (47) is positive, zero, or negative. Notice that (47) becomes

$$\hat{\sigma}(m, n) = 3 - m - n, \quad D = 3, \quad (48)$$

$$\hat{\sigma}(m, n) = 4 - 2m, \quad D = 4, \quad (49)$$

in three and four dimensions respectively. Thus in three dimensions three interactions³ are non-irrelevant, while there are infinitely many in four dimensions. Labelling interactions by a pair (m, n) we have:

D	(m,n)	relevant	marginal
3	(1,1)	(1,2), (2,1)	
4	(1,n)	(2,n)	

The conclusion is of course that the theory is super-renormalizable in three and renormalizable in four dimensions. In the BPHZ scheme this leads to the complication that the renormalizable case has infinitely many divergent graphs. In the present approach (with no divergent graphs) the renormalizable case will not be more complicated than the super-renormalizable one.

Consider a general difference equation of the form

$$L^\sigma f\left(\frac{p}{L}\right) - f(p) = g(p) \quad (50)$$

for functions on \mathbb{R}^N . Assume that $L > 1$, σ is integer valued, and $g \in \mathcal{C}^\infty(\mathbb{R}^N)$. We distinguish two cases, the irrelevant case with $\sigma < 0$ and the relevant case with $\sigma > 0$.

Let $\sigma < 0$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n)$.

1) The series

$$f(p) = - \sum_{m=0}^{\infty} L^{m\sigma} g\left(\frac{p}{L^m}\right) \quad (51)$$

is uniformly convergent on compact subsets of \mathbb{R}^N . It defines a function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$.

2) The function f given by (51) is a solution to (50). It is unique in the space of smooth functions.

3) Let $|g|_{\infty, \epsilon} < \infty$.⁴ Then f satisfies the bound

$$|f|_{\infty, \epsilon} \leq \frac{1}{1 - L^\sigma} |g|_{\infty, \epsilon}. \quad (52)$$

³Interaction here refers to the full kernel and comprises its value as well as derivatives at zero momentum.

⁴Recall that $|g|_{\infty, \epsilon} = \sup_{p \in \mathbb{R}^N} |g(p)| \exp(-\epsilon|p|)$.

If the derivatives of g are $L_{\infty,\epsilon}$ -bounded, so are the derivatives of f .

Formula (51) is obtained by iteration (50). Its convergence follows from the Weierstrass condition for uniform convergence [WW86]. It is checked to satisfy (50). The difference of two solutions to (50) satisfies the homogeneous equation $L^\sigma f\left(\frac{p}{L}\right) - f(p) = 0$. When σ is negative its only smooth solution is zero. The $L_{\infty,\epsilon}$ -bound is elementary.

Thus eq.(50) has been solved in the irrelevant case. In the relevant case the trick is to reduce the degree of relevancy by taking derivatives. Deriving (50) once, we have

$$L^{\sigma-1} \frac{\partial f}{\partial p} \left(\frac{p}{L} \right) - \frac{\partial f}{\partial p}(p) = \frac{\partial g}{\partial p}(p). \quad (53)$$

A partial derivative therefore reduces powercounting by one unit. The value of the derivative at the origin is obtained by evaluation of (53). The strategy is therefore a Taylor expansion with remainder term to an order where the remainder becomes irrelevant.

Let $\sigma \geq 0$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then

$$f(p) = \sum_{|\alpha| \leq \sigma} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial p^\alpha}(0) p^\alpha + \sum_{|\alpha| = \sigma+1} \frac{1}{\alpha!} \int_0^1 dt (1-t)^\sigma \frac{\partial^{|\alpha|} f}{\partial p^\alpha}(tp) p^\alpha \quad (54)$$

is a solution to (50) if and only if

$$\left(L^{\sigma-|\alpha|} - 1 \right) \frac{\partial^{|\alpha|} f}{\partial p^\alpha}(0) = \frac{\partial^{|\alpha|} g}{\partial p^\alpha}(0), \quad |\alpha| \leq \sigma, \quad (55)$$

$$L^{\sigma-|\alpha|} \frac{\partial^{|\alpha|} f}{\partial p^\alpha} \left(\frac{p}{L} \right) - \frac{\partial^{|\alpha|} f}{\partial p^\alpha}(p) = \frac{\partial^{|\alpha|} g}{\partial p^\alpha}(p), \quad |\alpha| = \sigma + 1. \quad (56)$$

The proof is obvious. As a consequence smooth solutions exist only when all marginal Taylor coefficients of g vanish at the origin. The marginal Taylor coefficients of the solutions are free parameters.

Let $\sigma \geq 0$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n)$.

1) Smooth solutions to (50) exist if and only if

$$\frac{\partial^{|\alpha|} g}{\partial x^\alpha}(0) = 0 \quad (57)$$

for all multi-indices with $|\alpha| = \sigma$.

2) In this case, we obtain a set of solutions which can be parametrized by the Taylor coefficients

$$c_\alpha = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) \quad (58)$$

for all multi-indices with $|\alpha| = \sigma$.

The proof is obvious. The grain of salt is that the marginal Taylor coefficients are not determined by eq. (50). The relevant Taylor coefficients and the irrelevant Taylor remainder are however explicitly known from (51), (55), and (56).

If the Taylor expansion (54) is pushed farther than to order σ , then also the Taylor coefficients with negative power counting obey (55). The iterative and the direct solution of (55) coincides when $\sigma - |\alpha| < 0$. Thus in terms of Taylor coefficients there is surprisingly no difference between the relevant and the irrelevant ones. Finally, remark that also the situation with non-integer scaling dimension σ has applications in the renormalization group. It is possible to cook up models with irrational scaling dimensions. Then the scheme is particularly simple because marginal coefficients are absent.

With this interlude we are essentially finished. The irrelevant kernels in (46) are solved by iteration. The non-irrelevant kernels are Taylor expanded. The Taylor coefficients are solved directly.

7 Marginalia

Marginal eigenvectors require a separate treatment. The following scheme is designed for ϕ^4 -theory in $D = 4$ dimensions. More generally it applies to models whose coupling is dimensionless. The underlying idea is that marginal eigenvectors possess a logarithmic flow. If also the coupling is marginal we may express the former logarithmic flow in terms of the latter. ϕ^4 -theory in $D = 3$ dimensions for instance calls for a different approach which will be explained elsewhere. There the coupling flows powerlike and one needs to introduce terms proportional to logarithms of the coupling to handle the marginal eigenvectors. For the rest of this section we put $D = 4$. Let us first do the computation of the β -function without wave function renormalization to explain the scheme in a slightly simpler setting. It is convenient to write it in terms of projectors on the marginal eigenvectors. Their meaning is nothing but Taylor expansion in momentum space. For a general Euclidean invariant functional

$$V(\phi) = \sum_{m=1}^{\infty} \frac{1}{2m!} \int d^D x_1 \dots d^D x_{2m} V_{2m}(x_1, \dots, x_{2m}) : \phi(x_1) \dots \phi(x_{2m}) : \quad (59)$$

define projectors

$$P_{4,0}V(\phi) = \frac{1}{4!} \int d^D x_1 : \phi(x_1)^4 : \int d^D x_2 d^D x_3 d^D x_4 V_4(x_1, x_2, x_3, x_4), \quad (60)$$

$$P_{2,2}V(\phi) = \frac{1}{2} \int d^D x_1 : (\partial\phi(x_1))^2 : \frac{-1}{D} \int d^D x_2 (x_1 - x_2)^2 V_2(x_1, x_2), \quad (61)$$

on the marginal eigenvectors

$$\mathcal{O}_{4,0}(\phi) = \frac{1}{4!} \int d^D x : \phi(x)^4 :, \quad (62)$$

$$\mathcal{O}_{2,2}(\phi) = \frac{1}{2} \int d^D x : (\partial\phi(x))^2 :. \quad (63)$$

These eigenvectors correspond to the particular kernels

$$V_{2n}(x_1, \dots, x_{2n}) = \delta_{n,1} \delta(x_2 - x_1) \delta(x_3 - x_1) \delta(x_4 - x_1), \quad (64)$$

$$V_{2n}(x_1, \dots, x_{2n}) = \delta_{n,1} \frac{-\partial^2}{\partial x_1^2} \delta(x_1 - x_2), \quad (65)$$

respectively, symmetrized in their entries. It is clear that they are reproduced by the projectors (60) and (61). Furthermore it is clear that the projectors satisfy

$$P_{4,0} \langle V \rangle_{\Gamma, \mathcal{S}\psi} = L^{4-D} P_{4,0} V(\psi), \quad (66)$$

$$P_{2,2} \langle V \rangle_{\Gamma, \mathcal{S}\psi} = P_{2,2} V(\psi). \quad (67)$$

The wave function term (63) is marginal (at order zero) in all dimensions and the ϕ^4 -vertex (62) in four dimensions. (61) and (60) are projectors on the marginal content of (59). Let us now consider the β -function. For all orders $n \geq 2$, the definition of the coupling constant implies that

$$P_{4,0} V^{(n)}(\phi) = 0. \quad (68)$$

That is the higher orders do not contain an eigenvector (63). Applying (61) to both sides of (41), and using (66) to conclude that the left side projects to zero, we find

$$\begin{aligned} b_n P_{4,0} V^{(1)}(\psi) &= \sum_{m=2}^{n-1} \frac{(-1)^{m+1}}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} \\ &\quad P_{4,0} \left\langle V^{(l_1)}; \dots; V^{(l_m)} \right\rangle_{\Gamma, \mathcal{S}\psi}^T. \end{aligned} \quad (69)$$

The projector $P_{4,0}$ selects the contributions built from lower order vertices, fluctuation propagators, and normal ordering propagators to the effective n 'th order ϕ^4 -vertex. This equation determines the coefficient b_n .

Let us now also introduce a wave function renormalization. This requires a slight modification of the renormalization scheme used so far. The second order condition on the wave function part is

$$b_2 P_{2,2} V^{(1)}(\psi) = -P_{2,2} \left\langle V^{(1)}; V^{(1)} \right\rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (70)$$

From it we see that we cannot leave the first order to remain a pure ϕ^4 -vertex (31) unless the right side of (70) does not contain a wave function term. This is unfortunately not the case. A wave function term is indeed generated to second (and arbitrary) order of perturbation theory. Therefore we have to include a wave function term already to first order in the running coupling and replace (31) by

$$V^{(1)}(\phi) = \frac{1}{4!} \int d^D x : \phi(x)^4 : + \frac{\zeta_1}{2} \int d^D x : (\partial\phi(x))^2 : . \quad (71)$$

Here ζ_1 is a new parameter, the first order wave function renormalization. Its value is determined by the second order equation (70). On a first inspection this equation looks quadratic in ζ_1 . This is not the case since

$$P_{2,2} \langle \mathcal{O}_{2,2}; \mathcal{O}_{2,2} \rangle_{\Gamma, \mathcal{S}\psi}^T = 0. \quad (72)$$

The reason is that the fluctuation propagator (1) is regular at zero momentum. Another fact which is pleasantly welcome is

$$P_{2,2} \langle \mathcal{O}_{2,2}; \mathcal{O}_{4,0} \rangle_{\Gamma, \mathcal{S}\psi}^T = 0. \quad (73)$$

As a consequence the first order wave function term does not alter the second order coefficient b_2 of the β -function. A wave function term on an external leg of the ϕ^4 -vertex makes the resulting amplitude vanish at zero momentum due to the regularity of (1) at zero momentum. As a consequence ζ_1 does not appear in

$$b_2 P_{4,0} V^{(1)}(\psi) = -P_{4,0} \langle V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (74)$$

from which b_2 is computed. Thus we can first find b_2 from (74) and then ζ_1 from (70).

Thereafter we can proceed with the computation of the full second order. This scheme generalizes to arbitrary order of perturbation theory. Assume that we have computed $V^{(m)}(\phi)$ for all orders $1 \leq m \leq n-1$ except for the wave function renormalization ζ_{n-1} at order $n-1$. Then the scaling equation

$$b_n P_{4,0} V^{(1)}(\psi) = -n P_{4,0} \langle V^{(1)}; V^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi}^T + \sum_{m=2}^{n-1} \frac{(-1)^{m+1}}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} P_{4,0} \langle V^{(l_1)}; \dots; V^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T \quad (75)$$

allows us to determine the n 'th order coefficient b_n of the β -function. Analogous to the second order case one does not find ζ_{n-1} on the right side of (75). To make the mechanism explicit let us split off the wave function term from the effective interaction and write

$$V^{(m)}(\psi) = \zeta_m \mathcal{O}_{2,2}(\psi) + W^{(m)}(\psi). \quad (76)$$

Then by the same reasons as above it follows that

$$P_{4,0} \langle V^{(1)}; V^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi} = P_{4,0} \langle W^{(1)}; W^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi} \quad (77)$$

is independent of ζ_{n-1} . But all other contributions in (75) stem from lower orders only. The final piece of gymnastics is to find ζ_{n-1} from

$$\begin{aligned} \frac{n(n-1)}{2} (b_1)^{n-2} b_2 P_{2,2} V^{(n-1)}(\psi) &= -b_n P_{2,2} V^{(1)}(\psi) - n b_1 b_{n-1} P_{2,2} V^{(2)}(\psi) - \\ &\sum_{m=2}^{n-1} \frac{1}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} \left(b_{l_1} \dots b_{l_m} V^{(m)}(\psi) + \right. \\ &\left. (-1)^m \langle V^{(l_1)}; \dots; V^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T \right) - n P_{2,2} \langle V^{(1)}; V^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi}^T + \\ &(-1)^{(n+1)} \frac{n(n-1)}{2} \langle V^{(1)}; \dots; V^{(1)}; V^{(2)} \rangle_{\Gamma, \mathcal{S}\psi}^T + (-1)^n \langle V^{(1)}; \dots; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T. \end{aligned} \quad (78)$$

Again the right side does not depend on the order $n-1$ wave function renormalization constant ζ_{n-1} because

$$P_{2,2} \langle V^{(1)}; V^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi} = P_{2,2} \langle W^{(1)}; W^{(n-1)} \rangle_{\Gamma, \mathcal{S}\psi}. \quad (79)$$

Notice that the wave function renormalization constant is hidden notationally on the left side of (78) in

$$P_{2,2} V^{(n-1)}(\psi) = \zeta_{n-1} \mathcal{O}_{2,2}(\psi). \quad (80)$$

Notice also that it comes together with pre-factors which do not vanish. The iterative scheme is now complete. In summary it goes as follows. To order n we first compute b_n from (75), then ζ_{n-1} from (78), and thereafter the effective interaction $V^{(n)}(\psi)$ except for ζ_n . Then we iterate the computation in the next order.

8 Estimates

In the iteration the lower order vertices are convoluted with hard and soft propagators to form the next order inhomogeneous scaling terms. In this section we present an estimate which shows that the integrals in this process converge. Consequently the expansion is finite to every order of perturbation theory. In a nutshell the expansion is finite, because the induction step involves perturbation theory with cutoff propagators and smooth vertices which do not grow too fast at large momenta. The fluctuation propagator has two sided cutoffs. The estimates

$$\|\hat{\Gamma}\|_{\infty, -2\epsilon} = \sup_{p \in \mathbb{R}^D} \left\{ \left| \hat{\Gamma}(p) \right| e^{2\epsilon|p|} \right\} < \infty, \quad (81)$$

$$\|\hat{\Gamma}\|_{1, -2\epsilon} = \int \frac{d^D p}{(2\pi)^D} |\hat{v}(p)| e^{2\epsilon|p|} < \infty, \quad (82)$$

for ϵ sufficiently small, should therefore cause no surprise. We save a small amount of exponential decrease to bound the large momentum growth of the vertices. It is then fairly obvious that a perturbation theory with this fluctuation propagator and $L_{\infty,\epsilon}$ -vertices is finite. The normal ordering propagator has an ultraviolet unit cutoff but no infrared cutoff. It satisfies

$$\|\hat{v}\|_{1,-2\epsilon} < \infty. \quad (83)$$

But this integrability in dimensions larger than two is sufficient to bound loop integrals with normal ordering propagator. Fortunately they never occur at external legs, where they would spoil the Taylor coefficients.

Recall that the induction step to order n involves the computation of b_n , ζ_{n-1} , and thereafter $V^{(n)}(\psi)$ (except for ζ_n). We present the argument for finiteness of $V^{(n)}(\psi)$ to some detail. Finiteness of the coefficients then follows by the same reasoning. Let us assume bounds on b_n , ζ_{n-1} , $V^{(n-1)}(\psi)$, and respective lower orders. The right side of the recursion equations (41) consists of two contributions, the *dynamical* contribution

$$\sum_{m=2}^n \frac{(-1)^m}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} \langle V^{(l_1)}; \dots; V^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T, \quad (84)$$

from the fluctuation integral, and the *kinematical* contribution from the flow of the coupling,

$$\sum_{m=1}^{n-1} \frac{1}{m!} \sum_{\substack{l_1, \dots, l_m \in \mathbb{N} \\ l_1 + \dots + l_m = n}} \binom{n}{l_1 \dots l_{m-1}} b_{l_1} \dots b_{l_m} V^{(m)}(\psi). \quad (85)$$

The second one immediately inherits a bound from the induction hypothesis on the lower orders. Recall that it arises from a re-organisation of the interaction as a polynomial of the bare coupling into a polynomial of the effective coupling. This re-organisation is finite because the lower order coefficients of the β -function are finite. To estimate the (84), we break it down to a sum of Feynman amplitudes. Fortunately we do not need more information besides that it can be written as a sum of finitely many amplitudes with certain properties. Each of the lower order interactions is a sum of vertices $V^{(l)}(\psi) = \sum_{k=1}^{l+1} V_{2k}^{(l)}(\psi)$ of the form

$$V_{2k}^{(l)}(\psi) = \frac{1}{(2k)!} \int d^D x_1 \dots d^D x_{2k} : \psi(x_1) \dots \psi(x_{2k}) : V_{2k}^{(l)}(x_1, \dots, x_{2k}). \quad (86)$$

We estimate individually each contribution in the truncated expectation value

$$\langle V^{(l_1)}; \dots; V^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T = \sum_{k_1=1}^{l_1+1} \dots \sum_{k_m=1}^{l_m+1} \langle V_{2k_1}^{(l_1)}; \dots; V_{2k_m}^{(l_m)} \rangle_{\Gamma, \mathcal{S}\psi}^T \quad (87)$$

for the sake of notational economy. Insert the expression (86). Then the task becomes an estimate for

$$\begin{aligned} \left\langle V_{2k_1}^{(l_1)}; \dots; V_{2k_m}^{(l_m)} \right\rangle_{\Gamma, \mathcal{S}\psi}^T &= \int \left(\prod_{j=1}^m \prod_{i=1}^{2k_j} d^D x_{j,i} \right) \prod_{j=1}^m V_{2k_j}^{(l_j)}(x_{j,1}, \dots, x_{j,2k_j}) \\ &\quad \left\langle \prod_{j=1}^m \left[: \prod_{i=1}^{2k_j} \phi(x_{j,i}) :; \right] \right\rangle_{\Gamma, \mathcal{S}\psi}^T. \end{aligned} \quad (88)$$

The truncated expectation value in (88) contains a product of clusters, each cluster being a normal ordered products of fields. The index j defines a colouring of the clusters. Eq. (88) is computed in two steps. Step one is

$$\begin{aligned} \left\langle \prod_{j=1}^m \left[: \prod_{i=1}^{2k_j} \phi(x_{j,i}) :; \right] \right\rangle_{\Gamma, \mathcal{S}\psi}^T &= \sum_{I_1 \subset \{1, \dots, 2k_1\}} \cdots \sum_{I_m \subset \{1, \dots, 2k_m\}} \prod_{j=1}^m : \prod_{i \in I_j} \mathcal{S}\psi(x_{j,i}) :_{v-\Gamma} \\ &\quad \left\langle \prod_{j=1}^m \left[: \prod_{i \in \{1, \dots, 2k_j\} \setminus I_j} \zeta(x_{j,i}) :_{\Gamma}; \right] \right\rangle_{\Gamma}^T. \end{aligned} \quad (89)$$

The truncated expectation value is here a standard one with mean zero. We sum over all subsets I_j of $\{1, \dots, 2k_j\}$. The interpretation is that for each field $\phi(x_{j,i})$ belonging to cluster j we decide whether it be a rescaled background field $\mathcal{S}\psi(x_{j,i})$ or a fluctuation field $\zeta(x_{j,i})$. The truncated expectation value is zero unless both the total number of fluctuation fields is even, and each cluster contains at least one fluctuation field. It is evaluated with Wick's theorem as a sum over pairings. The normal ordering forbids pairings within clusters. Step two is a re-normal ordering of the background fields. This is done with the help of

$$\begin{aligned} \prod_{j=1}^m : \prod_{i \in I_j} \mathcal{S}\psi(x_{j,i}) :_{v-\Gamma} &= \sum_{J_1 \subseteq I_1} \cdots \sum_{J_m \subseteq I_m} : \prod_{j=1}^m \prod_{i \in J_j} \mathcal{S}\psi(x_{j,i}) :_{v-\Gamma} \\ &\quad \left\langle \prod_{j=1}^m : \prod_{i \in J_j^c} \phi(x_{j,i}) :_{-v} \right\rangle_{-v}. \end{aligned} \quad (90)$$

The Gaussian expectation value is here un-truncated and with negative covariance. It is defined by Wick's rule as a sum of all pairings. The number of fields being contracted is again required to be even. The expectation value is zero for an odd number of fields. The sums in (90) go over all subsets J_j of I_j including the empty set and I_j itself. Their meaning is that for each field in cluster j we decide whether it be a truly external field or contracted with a field in another cluster $j' \neq j$. We can think of this process as a generation of additional loops with normal ordering covariance in every term generated

by the fluctuation integral. Having done both the truncated fluctuation integral and the un-truncated normal ordering "integral" we obtain

$$\begin{aligned} \left\langle V^{(l_1)}; \dots; V^{(l_m)} \right\rangle_{\Gamma, \mathcal{S}\psi}^T &= \sum_{p=1}^{\sum_{j=1}^m (k_j-1)+1} \frac{1}{(2p)!} \int d^D y_1 \dots d^D y_{2p} \\ &\quad L^{p(D+2)} K_{2p}^{(l_1, \dots, l_m)}(Ly_1, \dots, Ly_{2p}) \end{aligned} \quad (91)$$

with kernels given by

$$\begin{aligned} K_{2p}^{(l_1, \dots, l_m)}(y_1, \dots, y_{2p}) &= (2p)! \sum_{I_1 \subset \{1, \dots, 2k_1\}} \dots \sum_{I_m \subset \{1, \dots, 2k_m\}} \sum_{J_1 \subseteq I_1} \dots \sum_{J_m \subseteq I_m} \\ &\quad \delta_{|J_1| + \dots + |J_m|, 2p} \int \left(\prod_{j=1}^m \prod_{i \in \{1, \dots, 2k_j\} \setminus J_j} d^D x_{j,i} \right) \prod_{j=1}^m \left(\frac{1}{(2k_j)!} V_{2k_j}^{(l_j)}(x_{j,1}, \dots, x_{j,2k_j}) \right) \\ &\quad \left\langle \prod_{j=1}^m \left[: \prod_{i \in \{1, \dots, 2k_j\} \setminus I_j} \zeta(x_{j,i}) :_{\Gamma}; \right] \right\rangle_{\Gamma}^T \left\langle \prod_{j=1}^m : \prod_{i \in I_j \setminus J_j} \phi(x_{j,i}) :_{-v} \right\rangle_{-v}. \end{aligned} \quad (92)$$

The external points on the right side are here understood to be renamed. Their order is of no importance since all kernels are symmetric. The terms on the right hand side depend on the cardinalities $|I_j|$ and $|J_j|$ only. We are now ready to do the estimates. The right hand side of (92) is a sum of terms coming from writing out explicitly the expectation values in terms of contractions, the Feynman amplitudes. Their precise form is of no importance here. We restrict our attention to the number of lines and loops and combinatoric factors and find

$$L = \frac{1}{2} \sum_{j=1}^m (2k_j - |J_j|), \quad (93)$$

$$L_H = \frac{1}{2} \sum_{j=1}^m (2k_j - |I_j|), \quad (94)$$

$$L_H^{tree} = m - 1, \quad (95)$$

$$L_S = \frac{1}{2} \sum_{j=1}^m (|I_j| - |J_j|). \quad (96)$$

Here L is the total number of lines, that is, factors of propagators. L_H is the number of hard lines of fluctuation propagators coming from the truncated expectation value. L_S is the number of soft lines from re-normal ordering. Each contraction is cluster connected in terms of hard lines. Therefore each contraction contains a tree of hard lines. This number of tree hard lines is L_H^{tree} . The difference of L and L_H^{tree} is the number of loop integrals. We are ready to do the estimate. Put $L_{1,-2\epsilon}$ -norms on all loop lines of hard and soft propagators. Put $L_{\infty,-2\epsilon}$ -norms on the hard tree lines. Put

$L_{\infty,\epsilon}$ -norms on all vertices. Each of the terms on the right side of (92) being bounded we find

$$\begin{aligned} \|\widehat{K}_{2p}^{(l_1, \dots, l_m)}\|_{\infty, \epsilon} &\leq (2p)! \sum_{|I_1|=0}^{2k_j-1} \cdots \sum_{|I_m|=1}^{2k_m-1} \sum_{|J_1|=0}^{|I_1|} \cdots \sum_{|J_m|=0}^{|I_m|} \delta_{|J_1|+\dots+|J_m|, 2p} \\ &\prod_{j=1}^m \left(\binom{2k_j}{|I_j|} \binom{|I_j|}{|J_j|} \frac{1}{2k_j!} \|\widehat{V}_{2k_j}^{(l_j)}\|_{\infty, \epsilon} \right) \|\widehat{\Gamma}\|_{\infty, -2\epsilon}^{L_H^{tree}} \|\widehat{\Gamma}\|_{1, -2\epsilon}^{L_H - L_H^{tree}} \|\widehat{v}\|_{1, -2\epsilon}^{L_S} \\ &(2L_H - 1)!!(2L_S - 1)!!. \end{aligned} \quad (97)$$

More sophisticated estimates will be presented elsewhere. Hence $\|\widehat{K}_{2p}^{(l_1, \dots, l_m)}\|_{\infty, \epsilon}$ is finite in momentum space. Furthermore all loop integrals converge, and the result is smooth in the external momenta. It can in particular be Taylor expanded to any desired order. But then we are done: the right hand side of the recursion relation is smooth and can therefore be solved as above. Thus the expansion is indeed finite to every order of perturbation theory.

9 Second order

The second order scaling equation (39) involves a truncated expectation value of two first order interactions. The first order interaction is given by eq. (71), including a ϕ^4 -vertex and a wave function term. The first order wave function constant will come out of this second order computation. The expectation value is

$$\langle V^{(1)}; V^{(1)} \rangle_{\Gamma, \mathcal{S}\psi}^T = \langle \mathcal{O}_{4,0}; \mathcal{O}_{4,0} \rangle_{\Gamma, \mathcal{S}\psi}^T + 2\zeta_1 \langle \mathcal{O}_{4,0}; \mathcal{O}_{2,2} \rangle_{\Gamma, \mathcal{S}\psi}^T + \zeta_1^2 \langle \mathcal{O}_{2,2}; \mathcal{O}_{2,2} \rangle_{\Gamma, \mathcal{S}\psi}^T. \quad (98)$$

Each contribution is best taken care of separately. The computation consists of three steps. The first step is the fluctuation integral. It generates contractions between the vertices and shifts the normal ordering covariance. The second step is the rescaling of the external field. It restores the invariant normal ordering covariance and rescales the kernels of the effective interactions. The third step is a re-normal ordering of the result. Re-normal ordering creates additional loops with normal ordering propagators. The final result is

$$\begin{aligned} \langle \mathcal{O}_{4,0}; \mathcal{O}_{4,0} \rangle_{\Gamma, \mathcal{S}\psi}^T &= \frac{1}{2} \int d^D x \int d^D y : \psi(x) \psi(y) : L^{2+D} \left(\frac{1}{3} \Gamma(Lx - Ly)^3 + \right. \\ &\quad \left. \Gamma(Lx - Ly)^2 u(Lx - Ly) + \Gamma(Lx - Ly) u(Lx - Ly)^2 \right) + \\ &\quad \frac{1}{4!} \int d^D x \int d^D y : \psi(x)^2 \psi(y)^2 : L^4 \left(3\Gamma(Lx - Ly)^2 + \right. \\ &\quad \left. 6\Gamma(Lx - Ly) u(Lx - Ly) \right) + \end{aligned}$$

$$\begin{aligned}
& \frac{1}{6!} \int d^D x \int d^D y : \psi(x)^3 \psi(y)^3 : 20L^{6-D} \Gamma(Lx - Ly), \quad (99) \\
2 \langle \mathcal{O}_{4,0}; \mathcal{O}_{2,2} \rangle_{\Gamma, \mathcal{S}\psi}^T &= \frac{1}{2} \int d^D x : \psi(x)^2 : L^2 \left(\int d^D y \Gamma(y) \frac{-\partial^2}{\partial y^2} \Gamma(y) + \right. \\
& \quad \left. 2 \int d^D y v(y) \frac{-\partial^2}{\partial y^2} \Gamma(y) \right) +
\end{aligned}$$

$$\frac{1}{4!} \int d^D x \int d^D y : \psi(x)^3 \psi(y) : 8L^2 \frac{-\partial^2}{\partial y^2} \Gamma(Lx - Ly), \quad (100)$$

$$\langle \mathcal{O}_{2,2}; \mathcal{O}_{2,2} \rangle_{\Gamma, \mathcal{S}\psi}^T = \frac{1}{2} \int d^D x \int d^D y : \psi(x) \psi(y) : 2L^{-2+D} \frac{-\partial^2}{\partial x^2} \frac{-\partial^2}{\partial y^2} \Gamma(Lx - Ly) \quad (101)$$

Here $u(x) = L^{-2+D} v(L^{-1}x)$ denotes a rescaled normal ordering covariance. The partial derivatives are meant to act on rescaled fluctuation propagators. The relevant part can be extracted by Taylor expansion of the Fourier transformed kernels. The inhomogeneous terms of the second order scaling equation are given by

$$\begin{aligned}
\widehat{K}_2^{(2)}(p_1, p_2) &= L^{-2(4-D)} \left(b_2 \zeta_1 p_1^2 + L^2 \zeta_1 (c_1 + 2c_2) + L^{2-D} \zeta_1^2 \widehat{\Gamma}(L^{-1}p_1) + \right. \\
& \quad \left. \frac{L^2}{3} \widehat{\Gamma} \star \widehat{\Gamma} \star \widehat{\Gamma}(L^{-1}p_1) + L^2 \widehat{\Gamma} \star \widehat{\Gamma} \star \widehat{u}(L^{-1}p_1) + L^2 \widehat{\Gamma} \star \widehat{u} \star \widehat{u}(L^{-1}p_1) \right), \quad (102)
\end{aligned}$$

$$\begin{aligned}
\widehat{K}_4^{(2)}(p_1, \dots, p_4) &= L^{-2(4-D)} \left(b_2 + 8L^{2-D} p_1^2 \widehat{\Gamma}(L^{-1}p_1) + \right. \\
& \quad \left. 3L^{4-D} \widehat{\Gamma} \star \widehat{\Gamma}(L^{-1}p_1 + L^{-1}p_2) + 6L^{4-D} \widehat{\Gamma} \star \widehat{u}(L^{-1}p_1 + L^{-1}p_2) \right) \quad (103)
\end{aligned}$$

$$\widehat{K}_6^{(2)}(p_1, \dots, p_6) = L^{-2(4-D)} \left(20L^{6-2D} \widehat{\Gamma}(L^{-1}p_1 + L^{-1}p_2 + L^{-1}p_3) \right), \quad (104)$$

symmetrized in their momenta. The sum of momenta is constrained to zero. Here \star denotes convolution in momentum space (times $(2\pi)^{-D}$). The constants stand for the convergent loop integrals

$$c_1 = \int \frac{d^D p}{(2\pi)^D} p^2 \widehat{\Gamma}(p)^2, \quad (105)$$

$$c_2 = \int \frac{d^D p}{(2\pi)^D} p^2 \widehat{\Gamma}(p) \widehat{u}(p). \quad (106)$$

The momentum space convolutions are conveniently computed with the help of the parameter representations

$$\widehat{\Gamma}(p) = \int_1^{L^2} d\alpha e^{-\alpha p^2} \quad (107)$$

$$\widehat{u}(p) = \int_{L^2}^{\infty} d\alpha e^{-\alpha p^2}. \quad (108)$$

The momentum space integrals become Gaussian and can be reduced to the two formulas

$$\begin{aligned} \int \frac{d^D p}{(2\pi)^D} e^{-\alpha_1 p^2 - \alpha_2 (p-q)^2} &= (4\pi)^{-\frac{D}{2}} (\alpha_1 + \alpha_2)^{-\frac{D}{2}} e^{\frac{-\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} q^2}, \quad (109) \\ \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} e^{-\alpha_1 p_1^2 - \alpha_2 p_2^2 - \alpha_3 (p_1 + p_2 - q)^2} &= (4\pi)^{-D} (\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3)^{-\frac{D}{2}} \\ &\quad e^{\frac{-\alpha_1 \alpha_2 \alpha_3}{\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3} q^2}. \quad (110) \end{aligned}$$

A welcome feature of this parameter representation is that the result depends only on the external momentum squared. The scaling equation for the coefficient b_2 in the β -function follows from evaluation at zero momentum. It is given by

$$b_2 = -L^{4-D} \left(3\hat{\Gamma} \star \hat{\Gamma}(0) + 6\hat{\Gamma} \star \hat{u}(0) \right). \quad (111)$$

Its four dimensional value with exponential cutoff is computed to

$$b_2 = \frac{-6 \log(L)}{(4\pi)^2}. \quad (112)$$

It follows that the second order flow on the renormalized trajectory in four dimensions is given by

$$\beta(g) = g - \frac{3 \log(L)}{(4\pi)^2} g^2 + O(g^3). \quad (113)$$

It follows that the model is asymptotically free in the infrared limit at weak coupling. The scaling equation for the first order wave function constant is found to be

$$\zeta_1 = \frac{-1}{b_2} \frac{\partial}{\partial(p^2)} \left(\frac{1}{3} \hat{\Gamma} \star \hat{\Gamma} \star \hat{\Gamma}(p) + \hat{\Gamma} \star \hat{\Gamma} \star \hat{u}(p) + \hat{\Gamma} \star \hat{u} \star \hat{u}(p) \right) \Big|_{p^2=0}. \quad (114)$$

The result in four dimensions is

$$\zeta_1 = \frac{-1}{18(4\pi)^2}. \quad (115)$$

There is one relevant coordinate left at second order, the quadratic kernel at zero momentum. It is directly determined by the scaling equation

$$\begin{aligned} \left(L^{2-2(4-D)} - 1 \right) \hat{V}_2^{(2)}(0,0) &= L^{-2(4-D)} \left(L^2 \zeta_1 (c_1 + 2c_2) + \right. \\ &\quad \left. \frac{L^2}{3} \hat{\Gamma} \star \hat{\Gamma} \star \hat{\Gamma}(0) + L^2 \hat{\Gamma} \star \hat{\Gamma} \star \hat{u}(0) + L^2 \hat{\Gamma} \star \hat{u} \star \hat{u}(0) \right) \quad (116) \end{aligned}$$

Doing these integrals as a last exercise the effective mass constant at second order in four dimensions turns out as

$$\widehat{V}_2^{(2)}(0,0) = \frac{1}{(4\pi)^4} \left(2 \log(2) - \log(3) - \frac{1}{36} \right). \quad (117)$$

This completes the computation of the second order renormalization constants. The full second order kernels are then given by summed scaled parameter integrals. The explicit computation of this irrelevant part will not be pursued further here. If this second order approximation is for instance used in a numerical simulation as renormalization group improved action, it is reasonable to approximate it by a momentum space Taylor expansion to some chosen degree of irrelevancy. Then this computation supplies the non-irrelevant part.

10 Conclusions

The standard renormalization scheme departs from a bare action. The renormalized trajectory is reached upon infinite iteration of block spin transformations. A further block spin transformation does no harm to a renormalized action. It merely generates a renormalized renormalization group flow on the renormalized trajectory. In view of this renormalization group flow on the renormalized trajectory we speak of running couplings. We should mention that the behavior of a renormalized action under the field theoretic renormalization group was used by Callan [C76] in his proof of the BPHZ theorem, a polished version of which was given by Lesniewski [L83]. By universality the trajectory can be approached from a variety of bare actions. The notion of bare action is not unique. Consider for instance again the ultraviolet limit of ϕ^4 -theory in four dimensions (say at negative coupling). The bare action is required to converge to the stable manifold of the trivial fixed point. But this stable manifold of irrelevant couplings is infinite dimensional. The approach presented here is free of bare ambiguities. It is designed upon equations for the renormalized action.

The renormalized trajectory can be viewed itself as a means to investigate ultraviolet and infrared limit of a Euclidean field theory. It is ideally suited to perform an infinite number of block spin transformations. A single block spin step on the trajectory translates to a (generally non-linear) transformation on the low dimensional space of running couplings. As a finite dimensional dynamical system it is comparatively easy to analyze. Consider for instance the ϕ^4 -trajectory in the formalism above. There a block spin transformation becomes a transformation of a running coupling g in terms of a step β -function $\beta(g) = b_1 g + b_2 g^2 + O(g^3)$ with coefficients $b_1 = L^{4-D}$ and $b_2 < 0$. Here $L > 1$ is the block scale and D is the Euclidean dimension. For $D = 4$ it follows that the flow on the ϕ^4 -trajectory is not asymptotically free at weak positive coupling in the ultraviolet limit. The running coupling shrinks under a block spin transformation. An infinite number of block spin transformations require either the renormalized coupling

to approach zero or the bare coupling (as is believed) to tend to infinity. Perturbation theory is in this sense renormalization group inconsistent for the ultraviolet limit in four dimensions. We remark that another non-trivial renormalization group fixed point on the renormalized trajectory would allow to perform the ultraviolet limit. It would albeit suffice to require an infinite number of inverse block spin transformations for the running coupling to diverge. Either way requires control of the renormalized trajectory outside a vicinity of the trivial fixed point. This question is believed to be settled in favour of triviality by Aizenmann and Fröhlich. See [FFS92] and references therein. However even in this situation of an inconsistency of perturbation theory in the ultraviolet limit perturbation theory for the ϕ^4 -trajectory at weak couplings is a well posed problem.

The computational scheme presented here is very flexible. It applies to any form of block spin transformations in the vicinity of a trivial fixed point. The expansion is iterative and built upon a fairly simple recursion relation. One can therefore expect it to be useful for accurate bounds on renormalized series. For instance tree decay of interactions is an immediate consequence. However the L_∞ -bound is already sharper than the estimates of Polchinski [P84], and the improvements by Keller, Kopper, and Salmhofer [KKS90], and by Hurd [H89]. It does not lead to logarithms in a renormalization scale. Other sophisticated expansion technology can be found in Rivasseau's textbook [R91] together with a set of references to the Paris school of renormalization. In particular Rivasseau's effective expansion seems to be related to the one considered here.

The question whether our approach makes sense beyond perturbation theory is a very interesting one. Whether renormalization group invariance plus certain initial conditions determine a renormalized trajectory beyond perturbation theory we do not know. It is certainly the case where the perturbation expansion converges. Last not least it would be highly desirable to develop iterative techniques for unstable manifolds of nontrivial fixed points. A clear presentation of the weak coupling case around the trivial fixed point could be part of this way.

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